A diagramatic solution of the NK model with inflation (loosely based on Williamson's book)

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1 Introduction

Here we extend the analysis in Williamson's book by plotting the several figures simultaneously. The left half of below represents actual outcomes in the presence of sticky prices, while the right half shows the flexible-price equivalent values (these have the superscript n to denote natural values). We assume that expected inflation is equal to the inflation target in the analysis that follows.[1]

Before any shocks, the economy is at point a so that flexible and sticky price outcomes are the same (the output gap is zero).

The flexible-price part of the model consists of the subplots (b), (e) and (i). The suplots labelled 'rotation' are introduced so that I can link different subplots; there is no new information contain in the rotations and they are just 45-degree lines. There are two production functions, which are identical. They are included so that we can determine the values of employment and under sticky prices.

The equilibrium in the flexible-price part of the model is independent of its sticky price version so the former can be solved first and this analysis is explained thoroughly in the book.

For the sticky-price version, there is no output supply schedule. In (a), output is determined by the intersection of the IS and the real interest rate set by the central bank (CB). Recall that the monetary policy instrument is the nominal interest rate but due to price stickiness, this implies (implicitly) that it can control the real interest rate, albeit in the short-run only.

The Phillips curve (PC) is an equation linking current inflation to expected future inflation and $x \equiv Y - Y_m$, the gap between actual and flex-price levels of output, respectively. The PC will only shift if there is a change in Y_m or i'.[2]

The key equations in the NK model are the IS and the Phillips curve. The former is written as

$$Y - Y_m = -\alpha(r - r^*)$$

r^* is often also called the natural rate of interest, which I sometimes write as r^n. There are different ways in which we can interpret this equation, all of which are valid

1. Current output is a function of the current real interest rate: $Y = -\alpha r$, while flexible-price output is a function of the natural rate of interest rate: $Y_m = -\alpha r^*$. Thus, $x = Y - Y_m$ is given by the equation above.
2. Whenever $r = r^*$, the output gap is equal to zero.

The Phillips curve is written as

$$i = bi' + a(Y - Y_m)$$

Here's a little complication. In the long run, we'd expect $x = 0$ so $i = bi'$ but we'd also expect inflation to be constant and equal to its target (some number) but unless $b = 1$ we have a situation of changing inflation in the long run (constantly decreasing if $b < 1$). However, if we re-interpret i as inflation relative to target, in the long run this value is then zero (as is i') and the consistency is re-gained.

[1] I did not include the labour market under sticky prices but it is easy to see what is happening there.

[2] If we were to add shocks to the Phillips curve, these would have an average value of zero but then we could analyse the impact of a positive shock, which would shift the PC upwards.

1.1 Why do we assume that the central bank controls the ouptut gap?

The IS can be re-written as

$$\begin{aligned} Y - Y_m &= -\alpha(r - r^*) \\ x &= -\alpha\,(r - r^*) \\ x &= -\alpha\,(R - i' - r^*) \\ x &= \underbrace{\alpha\,(i' + r^*)}_{\equiv A} - \alpha R \end{aligned}$$

Think of A as a number, whose value is observed by the central bank. In that case, it can set R to be any number it likes in order to achieve a specific value of x. Therefore, as a short-cut we can assume that the monetary authority can directly control x and the value taken by A does not alter this. One implication is that if we are considering a policy maker whose objectives are to achieve certain values for x and i, the IS is irrelevant; we only need consider the Phillips curve. As a result, shocks to the IS (part of the A block) have no effects on x and i under an optimal policy. But to re-iterate, this only applies where we assume that the policy maker is trying to achieve $x = i = 0$.

Now we can consider the effects of different shocks on the model.

2 A sudden decrease in the nominal interest rate by the central bank

Hopefully, even before attempting to solve the model you will realise that this is a change in a nominal variable so that the effects on the flex-price version of the model are zero. Nothing changes to the natural values of those variables.

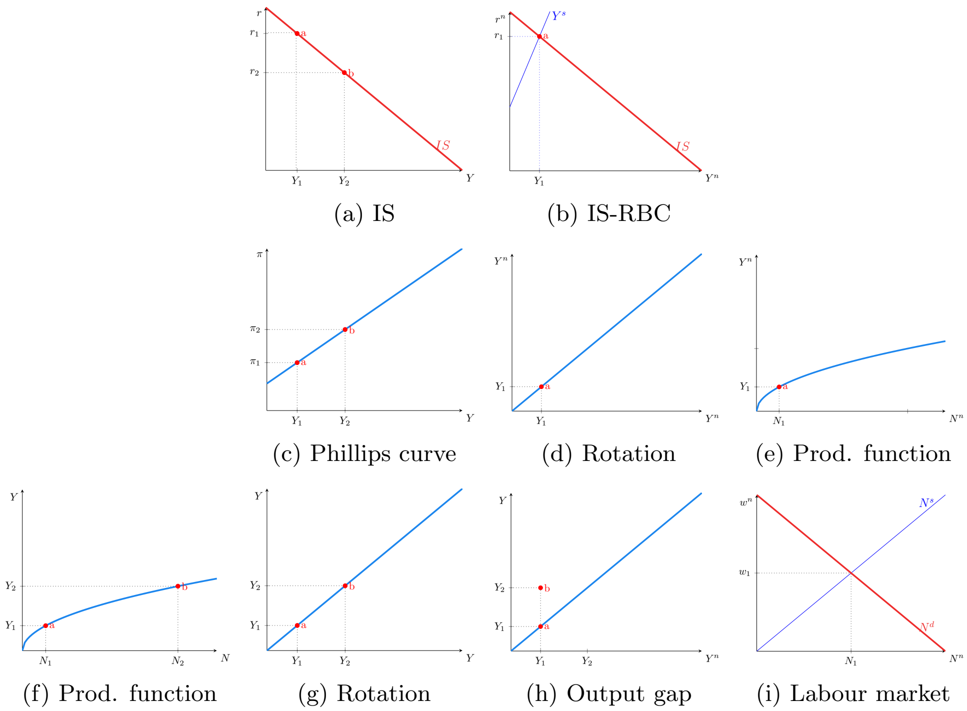
Turning to the sticky-price side of the model, a decrease in the nominal rate of interest (R) is accompanied by a one-for-one decrease in the real rate of interest. Therefore, the decrease in R

- lowers the real interest rate and as panel 1a shows, output increases (due to a rise in consumption and investment);
- this increase in output leads to an increase in inflation (panel 1c). Y_m has not changed but Y has increased so this leads to a higher rate of inflation (no shift in the PC) from the inflation target of π_1 to π_2.
- nothing has happened to the production function so the increase in output is obtained by increases in employment. Households are always on their labour supply but firms are not on their labour demand schedule. Instead, having set the prices for their goods, they will produce whatever is demanded, which may be a larger/smaller amount than what they had anticipated. But in order to produce this amount of output, they will have to pay a wage to obtain the labour they require. Put differently, if the demand for output rises to Y_2, we can see from the production function (f) that firms need to employ labour to amount to N_2. Had we plotted a diagram in (N, w) space showing labour supply (not shown), the only way firmst can obtain N_2 is via a higher wage.
- the monetary policy action affected the real interest rate only because of the sticky-price assumption but the flexible-price version of the model (the natural values) remain unaffected. Hence, the right half of the figure is unchanged. As a result, Y^n remains at a but Y has risen: output is greater than its flexible price counterpart. In other words, monetary policy is non-neutral.

Panel (h) just solves for the output gap: having obtained the values of Y and Y_m, we find where in that panel the economy is: points above it imply $Y > Y_m$ (economic expansion) and so on.

In summary, an unexpected temporary decrease in the nominal interest rate leads to an increase in output and inflation but leaves the flexible-price level of output unaffected. You can easily work out the effects on wages, employment, etc. Note also that although this is not obvious from the figure, this effect is temporary; in the next period the economy returns to point a, with the sticky and flexible-price versions of the model producing the same outcomes.

Figure 1: An unexpected decrease in the nominal interest rate



(a) IS (b) IS-RBC

(c) Phillips curve (d) Rotation (e) Prod. function

(f) Prod. function (g) Rotation (h) Output gap (i) Labour market

3 A shock to the natural rate of interest

Consider a shock, such as an increase in current TFP, that shifts output supply downwards so that both the natural rate of interest falls and Y^n rises. The changes in the natural values/rates are presented by the move from a to b in the figure.

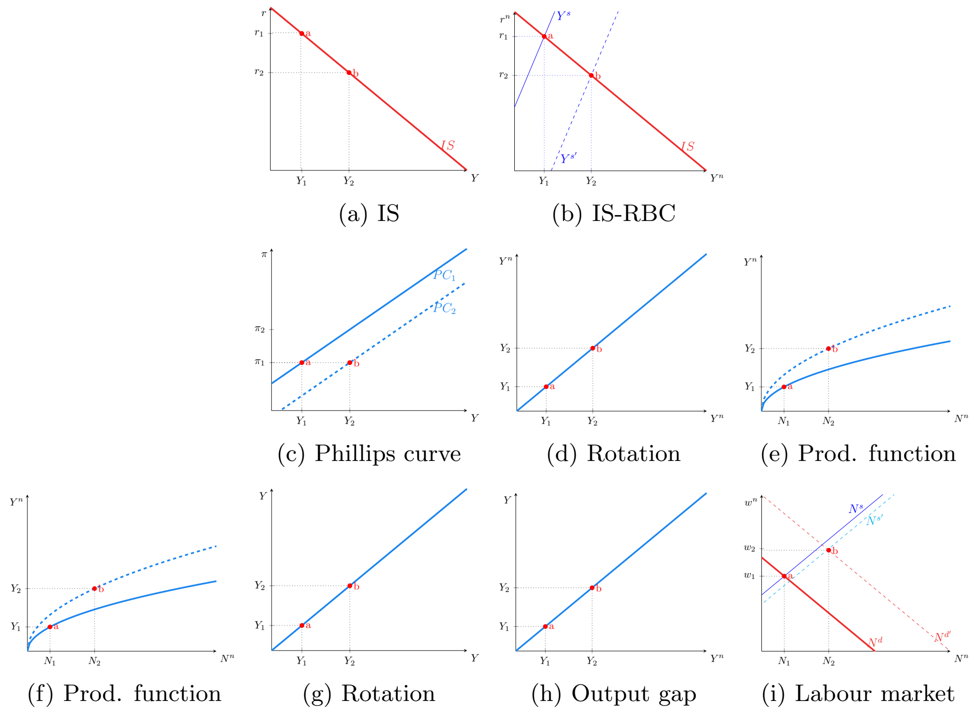
Again, the flex-price version of the model is solved in the book in detail.

- The flex-price side of the model shows that the increase in output and fall in the interest rate are accompanied by an increase in employment (both supply and demand increase, driven by the latter). The demand for labour shifted as this is given by the slope of the production function; labour supply has shifted up as the interest rate has decreased, which produces a contraction in labour supply.

Turning to the sticky-price version of the model, everything hinges on how the central bank reacts to the shock. Let us first assume that the central bank ensures that $r = r^n$.

- As $r = r^n$, the IS equation already tells us that $Y = Y_m$ so the output gap is zero. In panel (a) we have moved from a to b.
- While the shock represents a move along the IS, the Phillips curve shifts to PC_2 as the natural level of output, Y_m has increased. At the same time, we know that $x = 0$ and with π' equal to zero by assumption, we already
- The model is missing one equation: that representing the behaviour of the central bank regarding the setting of the nominal interest rate. Consequently, the analysis that follows is contingent on an 'if then' sequence.
- If the central bank does not change the nominal interest rate (and hence the real rate), equilibrium

Figure 2: A shock to the natural rate of interest



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in the IS is given by (Y_1, r_1). In other words, output does not change (so that it is below its flex-price level; a recession).

- Given the above, we have the same level of output but the Phillips curve has shifted down: inflation is now below its target, given by the intersection of Y_1 and the new Phillips curve.
- From the production function, employment is below its natural level.
- Lastly, from panel h, $Y < Y^n$.

4 Optimal policy

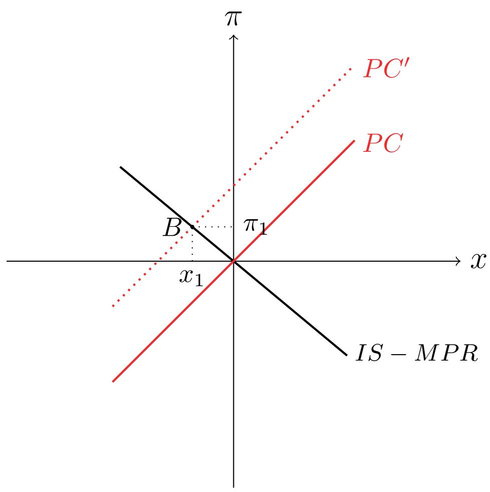
We concluded earlier that under optimal policy only the PC matters and the IS can be ignored. In addition, we can solve the model by assuming that the central bank chooses (temporarily) the value of x. In other words, given the Phillips curve, the central banker optimises by choosing the $(x,\ i)$ combination that she deems best. While ideally this would be for values (0, 0), such an outcome would only be attainable if the Phillips curve is going through $x = 0$ and $i = 0$, otherwise some alternative value will have to be made that balances the costs of each deviation.

Diagramatically, as the only equation that the central bank has to abide by is the PC, we can re-express are more into a very simple set-up. Rewrite the PC as

$$i = bi' + a(Y - Y_m)$$
$$i = bi' + ax$$

The intercept is bi', which in the absence of shocks is equal to zero. This is just an upward-sloping line going through the origin and with a slope of a.

Figure 3: Optimal policy in the NK model



- In the absence of shocks, the policy maker will choose $i = x = 0$.
- In the presence of a positive shock ($bi' > 0$), the PC will shift upwards. In this case, the point above will no longer be feasible and the policy maker will then set $x < 0$ and $i > 0$. However, whether most of the shock is absorbed by x or by i will be a function of the policy maker's objectives. If her concern is primarily inflation stabilisation, inflation will rise by a small amount while output will contract relatively more.
- If we were to simulate all these hypothetical shifts of the PC arising from shocks, we would then obtain the central bank's optimal response, which would be a downward sloping line going through (0, 0). It would be very flat is the concern for inflation stabilisaiton is much larger than that for stabilising x.
- The model then just consists of the PC and the IS-MPR line shown in Figure 3.

Figure 3 shows the impact of a shock to the PC. Starting at the origin, the shock poses a trade-off for the policy maker, whose best response is point B, implying inflation above target and $x < 0$.

Notice that with constant preferences (so the IS-MPR line does not shift) and with the PC being one of the building blocks of our model

- The relationship between x and inflation is negative. In other words, recessions ($x < 0$) are accompanied with above-average inflation.
- If there are no shocks to Y_m, then movements in x are the same as movements in Y: the relationship between output and inflation is negative, despite the Phillips curve seemingly implying that the relationship is positive.
- Therefore, if one finds a negative relationship between inflation and output in the data, this is not evidence that the Phillips curve is a myth or that it is a poor explanation of actual economies but it could just indicate that the central bank is succeeding in aiming to stabilise inflation and x.

5 The NK model with a Taylor rule

In order to determine the equilibrium values of x and inflation we need to know how the central bank responds to shocks. Under the optimal policy discussed above, policy is given by the IS-MPR schedule. An alternative is to assume some interest rate rule followed by the central bank, such as the Taylor rule (TR). The original TR has nominal interest rates responding positively to deviation of inflation and the output gap from their target levels, such as

$$R = \bar{R} + \phi(\pi - \bar{\pi}) + \mu x$$

so the long-run value of the nominal interest rate is $\bar{R} = r^* + \bar{\pi}$. For example, starting at $R = \bar{R}$, $\pi = \bar{\pi}$ and $x = 0$, if inflation is suddenly one percentage point above its target, $\pi - \bar{\pi} = 1$, the nominal interest rate will rise by ϕ. Of course, such an increase in the nominal interest rate will also affect x but here we are only discussing the direct impact. In general, for the model to provide a unique equilibrium we require $\phi > 1$ so that in response to an increase in inflation the rise in the nominal interest rate is such that real interest rates increase, which will then contract demand and dampen inflation.

There are two main reasons that Taylor-type rules are so widely used in academic research

1. It provides a reasonably good fit of actual interest rate setting by many central banks.
2. Such rules perform well across a wide range of models. In contrast, the IS-MPR schedule obtained above is model-specific and using such a rule in a different model may result in poor outcomes.

However, note that no central bank really follows a Taylor or Taylor-type rule; it is just a simple way of describing monetary policy that is often regarded as a reasonable approximation.

One difficulty in analysing our model with an interest rate rule is that we can no longer ignore the IS nor assume that the central bank controls the x directly. Our model is now given by

$$x = -\alpha(r - r^*) \tag{1}$$

$$i = i' + ax \tag{2}$$

$$R = r + i' \tag{3}$$

$$R = \bar{R} + \phi(i - \bar{i}) \tag{4}$$

Where now i is the inflation rate (not its deviation), $\bar{i}$ is the inflation target and $\bar{R}$ is the long-run value of the nominal interest rate, set by the central bank. The first equation is the IS and that is unchanged on the earlier analysis. Equation (2) is the Phillips curve where we have set $b = 1$; equation (3) is the Fisher equation and lastly, (4) is our monetary policy rule: we assume that the central bank increases the nominal interest rate (relative to $\bar{R}$) whenever inflation is above target. Note that as we are doing a 'what if' scenario, we could use any interest rate rule we wanted but the formulation here captures the key features of the Taylor rule in a manner that is as simple as possible (hence no output gap in this equation).

To understand our model, first consider its steady state properties. r^* is determined by the real factors and is independent of monetary policy/price stickiness so we can treat it as exogenous with a steady state value of $\bar{r}^*$. $\bar{R}$ is set by the central bank; we shall see below what this implies.

In the long run, the real interest rate r equals its natural rate $\bar{r}^*$, so that the first equation then implies that $x = 0$. Using this in the Phillips curve we find that in the long run $i = i' = \bar{i}$. This make sense: in the steady state, the inflation rate will be constant, but we have not yet determined what this value will be, only that it will not be changing over time. Turning to the Fisher equation, we have

$$\begin{aligned} R &= r + i' \\ \bar{R} &= \bar{r}^* + \bar{i} \\ \bar{i} &= \bar{R} - \bar{r}^* \end{aligned}$$

The last equation gives the solution for the inflation target: it equals the steady state nominal interest rate net of the natural rate of interest. Thus, the central bank controls the inflation target via its steady state value of R.

Now we can turn to our short-run analysis. We cannot directly put the above equations into a single diagram but we want to summarise everything into two curves: a Phillips curve and the interation between the interest rate rule (Taylor rule for short) with the IS. Therefore, we want two equations containing i and x.

Step 1: Combining the Fisher equation with the Taylor rule we have

$$r + i' = \bar{R} + \phi\,(-\bar{i}) \quad \Rightarrow r = \bar{R} + \phi\,(i - \bar{i}) - i'$$

Step 2: Substitute this into the IS

$$\begin{aligned} x &= -\alpha\left[\bar{R} + \phi(i - \bar{i}) - i' - r^*\right] \qquad \bar{R} - \bar{r}^* = \bar{i} \\ x &= -\alpha\left[\phi\,(i - \bar{i}) - (i' - \bar{i}) - (r^* - \bar{r}^*)\right] \end{aligned}$$

Re-write this equation in terms of i so that we can plot it

$$i = \bar{i} + \underbrace{\frac{1}{\phi}\left[(i' - \bar{i}) + (r^* - \bar{r}^*)\right]}_{B} - \frac{1}{\alpha\phi}x \tag{5}$$

This is a negative linear relationship between i and x with a slope equal to $-1/(\alpha\phi)$. The intercept is given by $\bar{i} + B$, which in the absence of shocks is just equal to $\bar{i}$.[3]

Our model is then given by the Phillips curve (2) and IS-TR (5). If we were to plot the latter it would look the same as the IS-MPR in Figure 3, going through the same intercepts but with two key exceptions:

1. The slopes are not necessarily the same. The interest rate rule is not derived from optimising behaviour so any difference in slope can be regarded as a deviation from optimality. While the IS-TR line in the figure is steeper than IS-MPR, this is just an example and it could have been flatter.
2. The IS-TR curve will shift whenever there are shocks to the IS, given by $r^* - \bar{r}^*$ and by deviations of expected inflation from the target; both of which are contained in B. However, the larger the value of ϕ, the strength of the response of interest rates to the inflation gap, the smaller the vertical shift.

Lastly, note that as ϕ goes to infinity, the equation (5) shows that $i = \bar{i}$: the line becomes horizontal going through $\bar{i}$ and the shocks to the IS are once again irrelevant for x and i. But then this comes at a cost: shocks to the PC are fully absorbed by x.

[3] By absence of shocks we mean that $i' = \bar{i}$ and $r^* = \bar{r}^*$.

Figure 4: Optimal policy vs Taylor rule in the NK model: a shock to the natural rate of interest

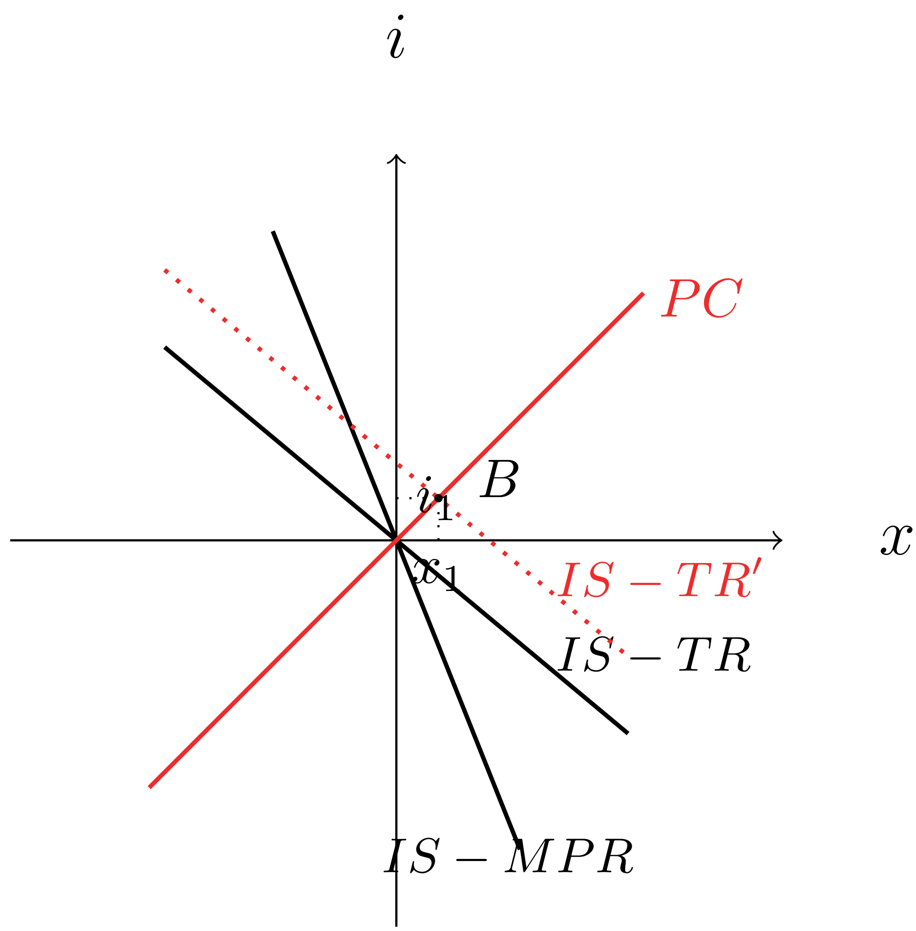


Figure 4 compares the effects of a shocks to the natural rate of interest, r^* under an optimal policy with a Taylor rule. The first thing to note is that the equilibrium at any point in time is given by the intersection of the Phillips curve with *one* IS-monetary policy rule, where the latter is only one of the two downward sloping lines. The two solid downward sloping lines are shown for comparison. I am assuming that the inflation target is zero so that pre-shock all lines go through the origin and note that in steady state the equilibrium is the same regardless of the monetary policy in place: both settings for monetary policy imply that inflation equals its target and $x = 0$. However, where the two policies differ is in how the shocks affect the economy.

In the diagram I have plotted the effects of a positive shock to r^*. This shock enters the IS only and we have already seen that under the optimal policy such shocks have no effects on inflation or output as they are fully offset by monetary policy. Therefore, with optimal policy the shock to r^* still leaves $i = x = 0$. In contrast, with a Taylor rule the increase in r^* causes an upward shift in the IS-TR but note that the magnitude of this shift depends (negatively) on ϕ. The intuition is as follows: a positive increase in r^* is expansionary as it puts upward pressure on $r - r^*$ and hence by raising x, the latter results in a higher i. However, as R rises in response to the increase in i, the overall change in $r - r^*$ is smaller but nonetheless positive, unlike the case with optimal monetary policy. Therefore, shocks to the IS move output, x and i in the same direction.

Lastly, note that with a Taylor-type rule, the relationship between x and i (or inflation and output) could go either way. In addition to being dependent on the model's parameters such as ϕ, if most of the shocks come the IS, the relationship is negative whereas if most of the shocks originate in the PC, the correlation will be negative. Therefore, a simple correlation is insufficient evidence in favour/against the presence of the Phillips curve in the data.